On the Beta-Inverted Weibull Distribution and it Application

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Abstract: this study focused on combining the inverted Weibull (IW) distribution and beta distribution with a view to obtain a distribution that is better than each of them individually in terms of the estimate of their characteristics and parsimonious in their parameters using a logit of beta (link function of the Beta generalized distribution by jones (2004)). The resulting model called Beta-inverted Weibull (BIW) distribution is better in terms of flexibility and shape. The statistical properties of the proposed distribution such as shape moments, moment generating function, asymptotic behaviorand hazard function were investigated. We carried out a practical application of BIW distribution on real life data and we discovered that the Beta inverted Weibull distribution has a better data representation than the inverted Weibull distribution.

Keywords: Beta inverted Weibull distribution, Shape moment, Moment generating function, Hazard function.

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1.0 INTRODUCTION

The inverted Weibull distribution is one of the most popular probability distribution to analyze the life time data with some monotone failure rates. Ref. [1] studied the properties of the inverted Weibull distribution and its application to failure data. Ref. [2] introduced the exponentiated Weibull distribution as generalization of the standard Weibull distribution and applied the new distribution as a suitable model to the bus-motor failure time data. Ref. [3] reviewed the exponentiated Weibull distribution with new measures. Ref. [4]studied the properties of transmuted Weibull distribution. We say that the random variable X has a standard inverted Weibull distribution (IWD) if its distribution function is given as:

$$G(x) = e^{-x^{-\beta}} \qquad x > 0, \quad \beta > 0 \tag{1}$$

The pdf of Inverted Weibull distribution is given by

$$g(x) = \beta x^{-\beta} e^{-x^{-\beta}} \tag{2}$$

The graph of the probability density function of the IW distribution is drawn below taking $b = \beta = 2.5$

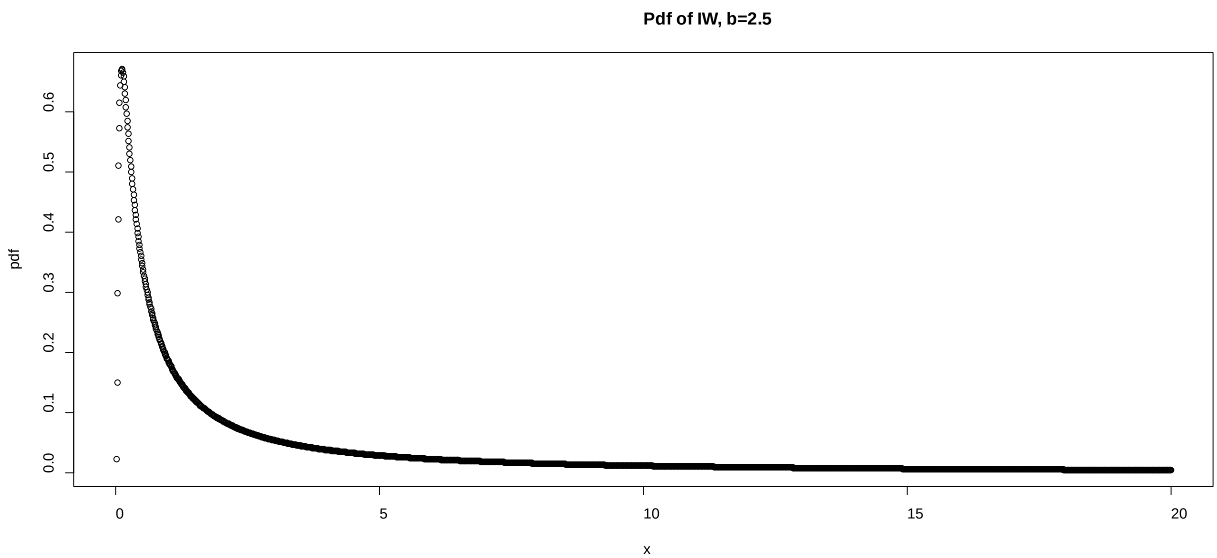

Figure 1. The graph of the pdf of IW distribution

2. THE PROPOSED BETA-INVERTED WEIBULL DISTRIBUTION

A new class of probability distributions which involves compounding beta family of distribution which include beta-normal (Eugene & Famoye, 2002) [5]; beta Gumbel (Nadarajah & kotz, 2004)[6], beta Weibull (Famoye. Lee & Olugbenga, 2005) [11], beta-Rayleigh (Akinsete & Lowe, 2009) [8]; beta-Laplace (Kozuboski & Nadarajah, 2008) [14]; beta pareto (Akinsete, Famoye & Lee, 2008) [7]; beta Gompertz Makeham (Chukwu & Ogunde, 2015)[13]; beta gamma, beta-t, beta-f, beta kumaraswamy among others.

Now let X be a random variable from the distribution with parameter defined in (1), using the logit of beta defined by Jones [9]as:

$$g(x) = \frac{1}{B(a,b)}[F(x)]^{a-1}[1 - F(x)]^{b-1}f(x) \tag{3}$$

And substituting (1) & (2) in (3) above, the beta-inverted Weibull distribution is derived as follows

$$g(x) = \frac{\beta}{B(a,b)}x^{-\beta}\left[e^{-x^{-\beta}}\right]^{a}\left[1 - e^{-x^{-\beta}}\right]^{b-1} \tag{4}$$

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Equation () the pdf of Beta-inverted Weibull becomes

$$g(x) = \frac{1}{B(a,b)}[P]^{a-1}[1-P]^{b-1}\frac{dP}{dx} \qquad (5)$$

The pdf of Beta-inverted Weibull distribution for different values of the parameters is given in figure below

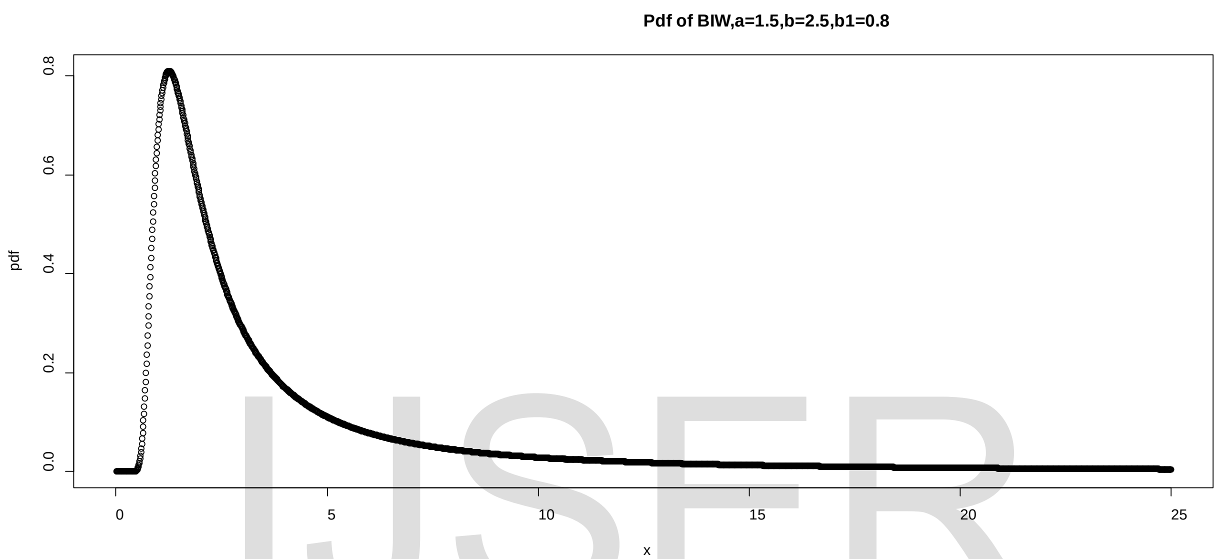

Figure 2. The graph of the pdf of BIW distribution.

3. Investigation of Proper pdf

Here we seek to investigate whether the proposed distribution integrate to unity.

$$\frac{1}{B(a,b)}\int_{0}^{1} \beta x^{-\beta}\left[e^{-x^{-\beta}}\right]^{a}\left[1-e^{-x^{-\beta}}\right]^{b-1}dx \qquad (6)$$

If we let

$$P = e^{-x^{-\beta}}$$

Such that $\frac{dP}{dx} = \beta x^{-\beta} e^{-x^{-\beta}}$, $dx = \beta x^{-\beta} e^{-x^{-\beta}} dP$

Substituting for dx in equation (6), we have

$$\frac{1}{B(a,b)}\int_0^1 \beta x^{-\beta}[P]^a[1-P]^{b-1}\frac{dP}{\beta x^{-\beta}e^{-x^{-\beta}}}$$

This result to

$$\frac{1}{B(a,b)}\int_0^1 [P]^{a-1}[1-P]^{b-1}dP = \frac{B(a,b)}{B(a,b)} = 1$$

4. Cumulative density function

The distribution of the beta inverted Weibull distribution is derived as follows:

$$P(X \leq x) = \frac{1}{B(a,b)}\int_0^x \beta t^{-\beta}\left[e^{-t^{-\beta}}\right]^a\left[1-e^{-t^{-\beta}}\right]^{b-1}dt \quad (7)$$

This reduces to

$$\frac{1}{B(a,b)}\int_0^X [P]^{a-1}[1-P]^{b-1}dP = \frac{B(P;a,b)}{B(a,b)} \quad (8)$$

As $P = e^{-x^{-\beta}}$

Finally,

$$G(x) = \frac{B\left(e^{-x^{-\beta}};a,b\right)}{B(a,b)} \quad (9)$$

Where $B(P;a,b)$ is an incomplete Beta function.

According to Jones (2004), the above expression can be written as:

$$G(x) = \frac{P^a}{B(a,b)}\left[\frac{1}{a} + \frac{1-b}{a+1}P + \cdots + \frac{(1-b)(2-b)(n-b)p^n}{n!(a+n)}\right] \quad (10)$$

The graph of the cumulative distribution function of Beta-inverted Weibull distribution for various values of the parameter is given in the diagram below

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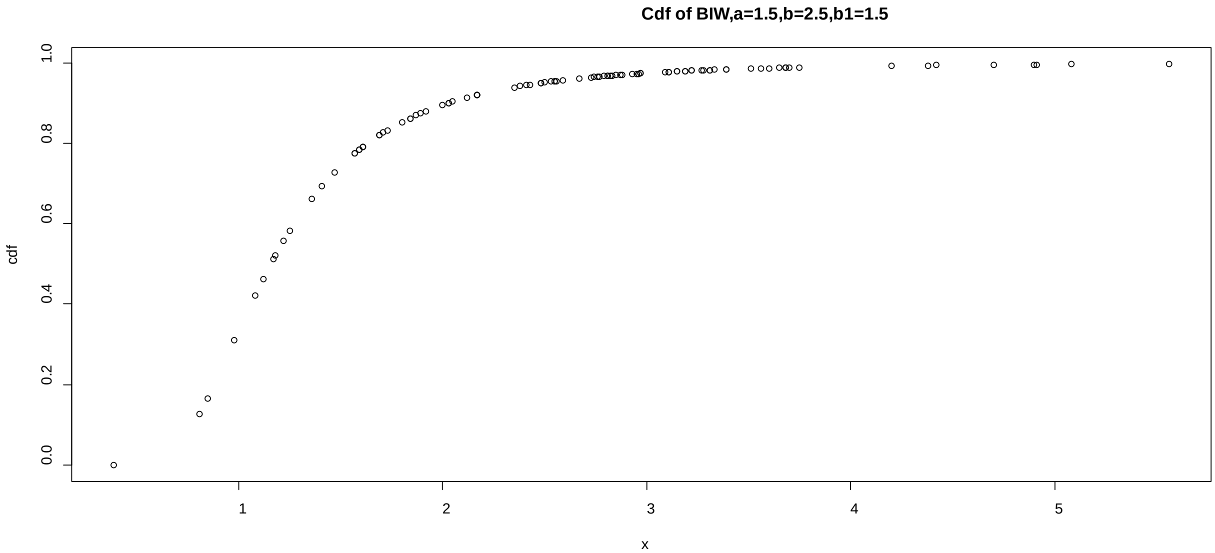

Figure 3. The graph of the cdf of BIW distribution

4.1 The Asymptotic Properties

In this section we investigate the properties of Beta-inverted Weibull distribution with a view to determine its performance as $x \longrightarrow \infty$. This can be done by investigating the limiting behaviour of the distribution.

$$\lim_{x \longrightarrow \infty} g(x) = \lim_{x \longrightarrow \infty} \frac{\beta}{B(a,b)} x^{-\beta} \left[e^{-x^{-\beta}}\right]^{a} \left[1 - e^{-x^{-\beta}}\right]^{b-1}$$

$$\lim_{x \longrightarrow \infty} g(x) = 0$$

We further investigate the CDF, G(x) as $x \longrightarrow 0$, we have that the $\lim G(x) \longrightarrow 0$, since $\lim_{x \longrightarrow 0} \beta x^{-\beta} \longrightarrow 0$

In literature, if as x tends to zero, PDF tends to zero and as $x \longrightarrow \infty$. It tend to zero, it is an indication that at least one mode exist. Therefore the Beta-inverted Weibull distribution has a mode.

5.0 The Hazard Rate Function

The hazard rate function of a random variable X with probability density function $g(x)$ and a cumulative distribution function G(x) is given by

$$h(x) = \frac{g(x)}{1-G(x)} \quad (11)$$

For the Beta-inverted Weibull distribution with $g(x)$and $G(x)$ respectively defined by (4) & (9). The hazard rate

function is expressed as

$$h(x) = \frac{\beta x^{-\beta}\left[e^{-x^{-\beta}}\right]^{a}\left[1-e^{-x^{-\beta}}\right]^{b-1}}{B(a,b) - B\left(e^{-x^{-\beta}};a,b\right)} \quad (12)$$

The above is the Beta-inverted Weibull mortality model.

When$a = b = 1$, the hazard function reduces to

$$h(x) = \frac{\beta x^{-\beta} e^{-x^{-\beta}}}{1-e^{-x^{-\beta}}} \quad (13)$$

The above expression is the Inverted Weibull mortality model.

6.0 Moment and Moment generating functions

Let X be a Beta-inverted Weibull random variable as given in (4). According to Hosking (1990), when a random variable X follows a generalized beta generated distribution that is $X \sim GBG(f,a,b,c), then\ \mu_r^1 = E(F^{-1}U^{\frac{1}{c}})^r$, where U$\sim B(a,b), c\ is\ constant\ and\ F^{-1}(x)$ is thye inverse of the CDF of the Beta-inverted Weibull distribution. Since BIW distribution is a special form for $c = 1$, we have the r^{th} moment of the Beta-generated distribution as

$$\mu_r^1 = \frac{1}{B(a,b)}\int_0^1 [F^{-1}(x)]^r\, x^{a-1} x^{a-1}(1-x)^{b-1}dx \quad (14)$$

$$\mu^1 = 1$$

The Taylor series expansion around the point $E\left(X_f\right) = \mu_f$ to obtain

$$\mu_r^1 \cong \sum_{k=0}^{r} \binom{r}{k} \left[F^{-1}(\mu_f)\right]^{r-k} \left[F^{-1(1)}\mu_f\right]^k \sum_{k=0}^{r} (-1)^i \binom{k}{i}$$

Cordeiro and de Castro (2011) [10]gave alternative series expansion for μ_r^1 in terms of $r(r,m) = E[Y^r F(Y)^m]$ where Y follows the parent distribution, then for $m = 0,1, \ldots$.

$$\mu_r^1 = \frac{1}{B(a,b+1)} \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} r(r,a,i-1)$$

Now using another moment generating function of X for generalized Beta distribution given by Cordeiro and de Castro (2011) [10] as

$$M_{(t)} = \frac{1}{B(a,b+1)} \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} \rho(t,ai-1) \tag{15}$$

Where

$$\rho(t,r) = \int_{-\infty}^{\infty} e^{tx} [F(x)]^r f(x) dx$$

Then

$$M_{(t)} = \frac{1}{B(a,b+1)} \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{ai-1} f(x) dx \tag{16}$$

The moment generating function of Beta-inverted Weibull distribution is obtained as

$$M_{(t)} = \frac{1}{B(a,b+1)} \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} \int_{-\infty}^{\infty} e^{tx} \left[e^{-x^{-\beta}}\right]^{ai-1} \beta x^{-\beta} e^{-x^{-\beta}} dx \tag{17}$$

By setting $a = b = i = 1$ in (17) gives the moment generating function of the Inverted Weibull distribution.

7. Estimation of Parameters

Let $x_1, x_2, \dots, x_n$ be a random variable distributed according to (4) the likelihood function of vector parameters given by $\theta(a, b, \beta)$ is

$$L(\theta) = \prod_{i=1}^{n}\left[\frac{\beta}{B(a,b)} x^{-\beta}\left[e^{-x^{-\beta}}\right]^{a}\left[1-e^{-x^{-\beta}}\right]^{b-1}\right] \qquad (18)$$

And its log likelihood is given as

$$l(\theta) = nlog(\beta) - \beta\log(x_i) - a\sum_{i=1}^{n} x_i^{-\beta} + (b-1)\sum_{i}^{n}\log\left(1-e^{-x^{\beta}}\right) - nlog[B(a,b)] \qquad (19)$$

Note that, $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a,b)}$, then the above equation will transform to

$$l(\theta) = nlog(\beta) - \beta\log(x_i) - a\sum_{i=1}^{n} x_i^{-\beta} + (b-1)\sum_{i}^{n}\log\left(1-e^{-x^{\beta}}\right) - nlog\Gamma(a) - nlog\Gamma(b) + nlog\Gamma(a+b) \qquad (20)$$

The score vector is obtained as follows

$$\frac{\partial l}{\partial a} = -\sum_{i=1}^{n} x_i^{-\beta} - \frac{n\Gamma(a)'}{\Gamma(a)} + \frac{n\Gamma(a+b)'}{\Gamma(a+b)} \qquad (21)$$

$$\frac{\partial l}{\partial b} = -\sum_{i=1}^{n}\log\left(1-e^{x_i^{-\beta}}\right) - \frac{n\Gamma(b)'}{\Gamma(b)} + \frac{n\Gamma(a+b)'}{\Gamma(a+b)} \qquad (22)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i^{-\beta} - \frac{n\Gamma(a)'}{\Gamma(a)} + \frac{n\Gamma(a+b)'}{\Gamma(a+b)} \qquad (23)$$

8. APPLICATION

In this section we illustrate the applicability of the Beta-Inverted Weibull distribution by considering a data set consists of 63 observations of the strengths of 1.5cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the units of measurement are not given in the paper. The data are:0.55,0.93,1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2 , 0.74, 1.04, 1.27, 1.39, 1.49, 1.53,1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69,1.76, 1.84, 2.24, 0.81, 1.13,1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78,1.89. Table 1.0 gives the exploratory data analysis of the data considered, Table 2. gives the maximum likelihood estimates of the parameters with their standard error in parenthesis and Table 3. gives the criteria for comparison. Figure 4 and figure 5. represents the boxplot diagram for the data empirical density and the cumulative density of the data considered.

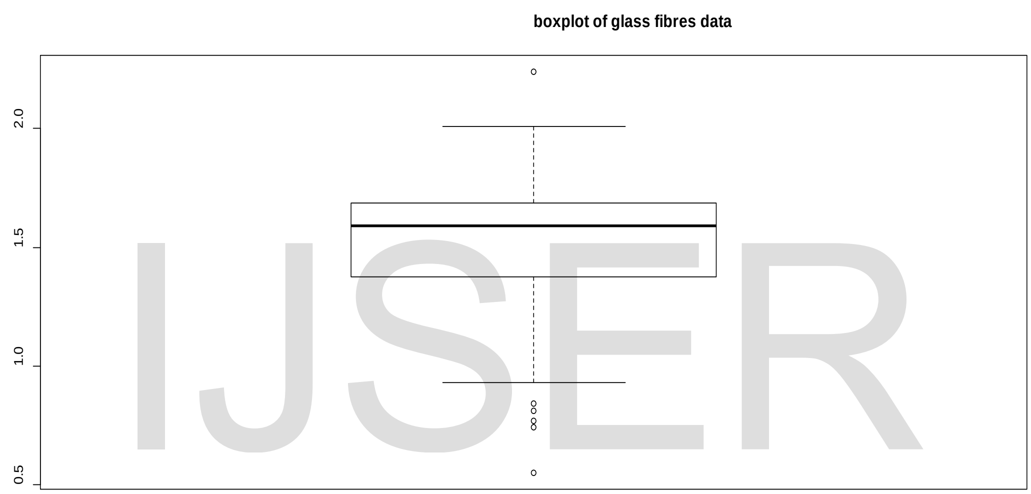

Figure 4. Boxplot for the glass fibre data

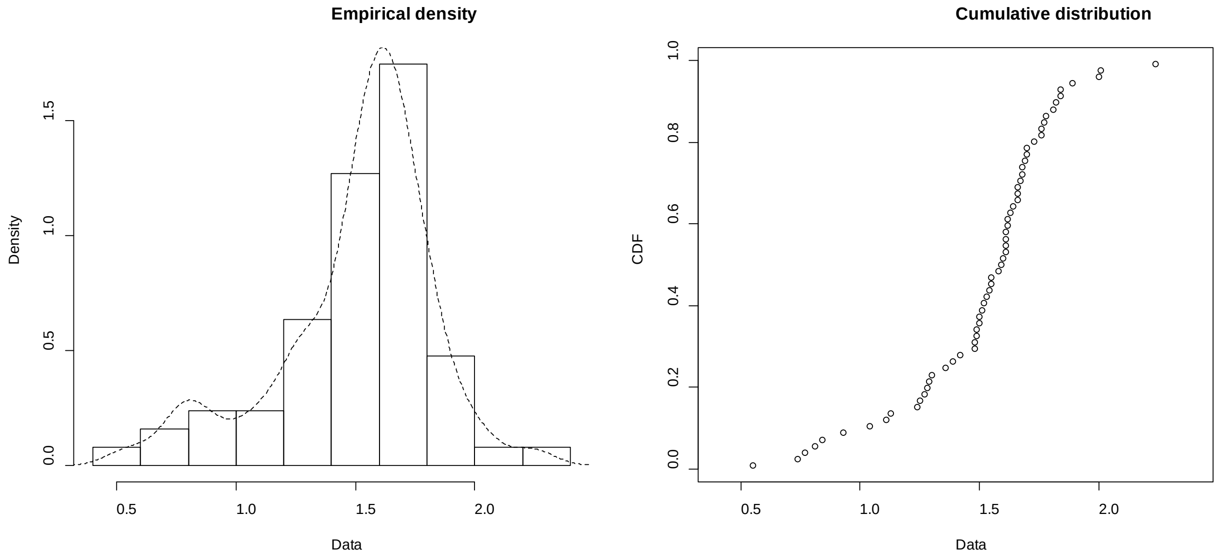

Figure 5. Graph of the empirical density and cumulative distribution function the glass fibre data

Table 1 Descriptive Statistics on Breaking stress of Carbon fibres.

Min	Q_1	Median	$mean$	Q_3	Max	$kurtosis$	Skewness
0.550	1.375	1.590	1.507	1.685	2.240	-0.922	1.103

Table2. MLEs(standard error in parenthesis) and the statistics $l(\hat{\theta})$, AIC, BIC and $HQIC$

$Model$	$Estimates$			K	PV
BIW (a, b, β)	9.412 (4.410)	0.268 (0.064)	13.909 (6.887)	0.1245	0.1741
$TIWD$ (λ, β)	0.7073 (0.399)	0.6902 (0.0574)	-	0.2439	0.00018
IW (β)	52.049 (0.047)	-	-	0.229	0.00054

Table 3. Measures of Goodness of fit

$Model$	$l(\hat{\theta})$	AIC	BIC	HQIC	CAIC	A^*	W
BIW (a, b, β)	-109.094	224.189	231.181	226.983	224.522	2.612	0.4302
$TIWD$ (λ, β)	-152.483	308.967	313.628	310.829	309.131	4.983	0.851
IW (β)	-154.278	310.556	312.887	311.487	310.610	5.269	0.9036

We employ the statistical tools for model comparison such as Kolmogorov-Smirnov (K-S) statistics, Anderson Darling statistic (AD), crammer von misses statistic (W), Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Hannan Quinine information criterion (HQIC) and Bayesian information criterion o choose the best possible model for the data sets among the competitive models. The selection criterion is that the lowest AIC, CAIC, BIC and HQIC correspond to the best fit model.

9. Conclusion:

The existing one parameter Inverted Weibull distribution is extended with the introduction of two extra shape parameters, thus defining the Beta-Inverted Weibull distribution which has a better shape and provides a better fit tpo the data in terms of the minimum values of the AIC, BIC, HQIC, CAIC, W and A^*

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